

A shape bias transformation with application to characteristic functions*

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Abstract

A probability transformation of a r.v. X with $EX^2 > 0$ and the characteristic function (ch.f.) $f(t) = \mathbb{E}e^{itX}$ is introduced. It is defined as arbitrary r.v. with the ch.f. $f^*(t) = -f''(t)/EX^2$. For any r.v. X with $EX = 0$, $EX^2 > 0$ and $\mathbb{E}|X|^3 < \infty$ the precise estimate $L_1(X^*, X) \leq \mathbb{E}|X|^3/EX^2$ is proved, where X^* is any r.v. with the ch.f. $f^*(t)$, $L_1(X^*, X)$ is the L_1 -distance between X and X^* . As a corollary, some new estimates for the proximity of the ch.f. $f(t)$ to the normal one are proved involving the double integrals of the corresponding transformation.

Key words and phrases: probability transformation, zero bias transformation, size bias transformation, characteristic function, L_1 -metric

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1 Introduction

Let X be a random variable (r.v.) with the distribution function (d.f.) $F(x) = \mathbb{P}(X < x)$, $x \in \mathbb{R}$, and the characteristic function (ch.f.) $f(t) \equiv \mathbb{E}e^{itX}$, $t \in \mathbb{R}$. As is well known, if X is nonnegative with $0 < EX < \infty$, then

$$\frac{f'(t)}{f'(0)}, \quad t \in \mathbb{R}, \quad (1)$$

is a ch.f., and if $0 < EX^2 < \infty$, then

$$\frac{f'(t) - f'(0)}{tf''(0)}, \quad \frac{f''(t)}{f''(0)}$$

are ch.f.'s as well (see, e.g., [13, Theorem 12.2.5]). The probability transformation given by (1) is called the *X-size bias transformation*. By a transformation of a random variable we mean that of its distribution. The *X-size bias transformation* was introduced by Goldstein and Rinott [5] for the purpose of estimation of the accuracy of the multivariate normal approximation to

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nonnegative random vectors under conditions of local dependence by Stein's method. Namely, in [5] an almost surely nonnegative r.v. $X^{(s)}$ with $0 < \mathbf{E}X < \infty$ is said to have the X -size biased distribution if

$$d\mathbf{P}(X^{(s)} < x) = \frac{x}{\mathbf{E}X} dF(x), \quad x \in \mathbb{R}. \quad (2)$$

It is easy to see that the distribution given by (2) has the ch.f. given by (1), hence, by virtue of the uniqueness theorem, definitions (1) and (2) are equivalent. In the same paper Goldstein and Rinott also noticed that the distribution of $X^{(s)}$ may be characterized by the relation

$$\mathbf{E}XG(X) = \mathbf{E}X\mathbf{E}G(X^{(s)}),$$

which should hold for all functions G such that $\mathbf{E}XG(X) < \infty$.

As regards the second transformation, if $\mathbf{E}X = 0$, then the distribution given by the ch.f.

$$\frac{f'(t) - f'(0)}{tf''(0)} = -\frac{1}{\sigma^2} \cdot \frac{f'(t)}{t},$$

where $\sigma^2 = \mathbf{E}X^2 > 0$, is called the X -zero biased distribution. This definition was introduced by Goldstein and Reinert in [3] in an equivalent form for the purpose of generalization of the size bias transformation to r.v.'s taking both positive and negative values and was inspired by the characteristic property of the mean zero normal distribution as the unique fixed point of the zero bias transformation. Namely, in [3] a r.v. $X^{(z)}$ is said to have the X -zero biased distribution if

$$\mathbf{E}XG(X) = \sigma^2 \mathbf{E}G'(X^{(z)})$$

for all absolutely continuous functions G for which $\mathbf{E}XG(X)$ exists. The zero biased transformation possesses the following elementary properties (most of them are noticed/proved in [3]):

1. The zero biased distribution is absolutely continuous and unimodal about zero with the probability density function

$$p(x) = \sigma^{-2} \mathbf{E}X \mathbb{I}(X > x), \quad x \in \mathbb{R},$$

and the ch.f.

$$\mathbf{E}e^{itX^{(z)}} \equiv -\frac{1}{\sigma^2} \cdot \frac{f'(t)}{t}, \quad t \in \mathbb{R}.$$

2. $X^{(z)} \stackrel{d}{=} X$ if and only if X has the normal distribution with zero mean [17, 3].
3. The zero biased transformation preserves symmetry.
4. $\sigma^2 \mathbf{E}(X^{(z)})^n = \mathbf{E}X^{n+2}/(n+1)$ for $n \in \mathbb{N}$, in particular, $\sigma^2 \mathbf{E}X^{(z)} = 0.5\mathbf{E}X^3$.
5. If $X = Y_1 + \dots + Y_n$, where Y_1, \dots, Y_n are independent r.v.'s with zero means and $\mathbf{E}Y_j^2 = \sigma_j^2 > 0$ so that $\sigma_1^2 + \dots + \sigma_n^2 = \sigma^2$, then $X^{(z)} = X_I + Y_I^{(z)}$, where I is a random index independent of Y_1, \dots, Y_n with the distribution $\mathbf{P}(I = i) = \sigma_i^2/\sigma^2$, $i = 1, \dots, n$, and $X_i = X - Y_i = \sum_{j \neq i} Y_j$.

6. The following non-trivial fact was proved in 2009 independently by Goldstein [2] and Tyurin [19, 18]:

$$L_1(X, X^{(z)}) \leq \frac{\mathbb{E}|X|^3}{2\sigma^2}, \quad (3)$$

$L_1(X, Y)$ being the L_1 -distance between the r.v.'s X and Y ,

$$L_1(X, Y) = \inf \left\{ \mathbb{E}|X' - Y'| : X' \stackrel{d}{=} X, Y' \stackrel{d}{=} Y \right\}, \quad \mathbb{E}|X| < \infty, \mathbb{E}|Y| < \infty.$$

As regards the third transformation, we give here the following

DEFINITION 1. Let X be a r.v. with the ch.f. $f(t)$ and $\sigma^2 \equiv \mathbb{E}X^2 \in (0, \infty)$. The distribution given by the ch.f.

$$f^*(t) \equiv -\frac{f''(t)}{\sigma^2}, \quad t \in \mathbb{R}, \quad (4)$$

is called the *X-shape biased distribution*.

It is easy to see that

1. A r.v. X^* has the X -shape biased distribution if and only if its d.f. $F^*(x)$ satisfies

$$dF^*(x) = \frac{x^2}{\sigma^2} dF(x), \quad x \in \mathbb{R}. \quad (5)$$

2. A r.v. X^* has the X -shape biased distribution if and only if for all functions G for which $\mathbb{E}X^2G(X)$ exists,

$$\mathbb{E}X^2G(X) = \sigma^2\mathbb{E}G(X^*). \quad (6)$$

3. $X^* \stackrel{d}{=} X$ if and only if $\mathbb{P}(|X| = \sigma) = 1$, i.e. any Bernoulli distribution with symmetric atoms is a fixed point of the shape bias transformation. This can be verified by noticing that the solution of the corresponding linear homogeneous differential equation $f''(t) + \sigma^2 f(t) = 0$ of the second order with the initial condition $f(0) = 1$ has the form $f(t) = pe^{i\sigma t} + (1-p)e^{-i\sigma t}$, $p \in \mathbb{R}$, being a ch.f. if and only if $p \in [0, 1]$.
4. $(X^*)^2 \stackrel{d}{=} (X^2)^{(s)}$, where $(X^2)^{(s)}$ has the X^2 -size biased distribution.
5. $(cX)^* = cX^*$ for any constant $c \in \mathbb{R}$.
6. $\sigma^2\mathbb{E}(X^*)^n = \mathbb{E}X^{n+2}$, for $n \in \mathbb{N}$, in particular, $\sigma^2\mathbb{E}X^* = \mathbb{E}X^3$.

Moreover, the following estimate for the proximity between X and X^* in terms of L_1 -metric will be proved in this paper.

THEOREM 1. If $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ and $\mathbb{E}|X|^3 < \infty$, then

$$L_1(X, X^*) \leq \mathbb{E}|X|^3,$$

moreover, for any $\varepsilon > 0$ there exists a distribution of a r.v. X concentrated in two points, such that $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 < \infty$, and

$$L_1(X, X^*) > (1 - \varepsilon)\mathbb{E}|X|^3.$$

The existence of the shape bias transformation follows from the earlier result of [13] mentioned above. Moreover, in 2005, Goldstein and Reinert [4] proved the existence of a class of transformations of probability distributions that is characterized by equations like (6). Namely, the authors described a class of measurable functions $T: \mathbb{R} \rightarrow \mathbb{R}$ that provide the existence and uniqueness of the distribution of a random variable $X^{(T)}$ such that

$$\mathbb{E}T(X)G(X) = \mathbb{E}G^{(m)}(X^{(T)}) \cdot \frac{\mathbb{E}X^m T(X)}{m!}$$

for all measurable m times differentiable functions $G: \mathbb{R} \rightarrow \mathbb{R}$ with $\mathbb{E}|T(X)G(X)| < \infty$. The authors also noticed that this class includes the zero- and size- bias transformations respectively with $m = 1$, $T(x) = x$ and $m = 0$, $T(x) = x^+$. It is easy to see that this class also includes the shape bias transformation (with $m = 0$, $T(x) = x^2$).

However, the properties of the shape bias transformation are formulated and proved here for the first time.

2 Motivation and applications

The zero bias transformation gives an opportunity to construct an integral estimate for the proximity of a ch.f. with zero mean to the normal one with the same variance in terms of the proximity of the corresponding zero biased distribution to the original one, which might be sharper than non-integral estimates based on the Taylor formula in the neighborhood of zero. Namely, for the sake of convenience put $\sigma^2 = 1$ implying $\beta_3 \geq 1$ by the Lyapounov inequality. Then using the elementary relations

$$\begin{aligned} f(t) - e^{-t^2/2} &= e^{-t^2/2} \int_0^t (f(u)e^{u^2/2} - 1)' du = e^{-t^2/2} \int_0^t (f'(u) + \sigma^2 u f(u)) e^{u^2/2} du = \\ &= e^{-t^2/2} \int_0^t (\mathbb{E}e^{iuX} - \mathbb{E}e^{iuX^{(z)}}) u e^{u^2/2} du, \quad (7) \end{aligned}$$

and the estimate for the difference of arbitrary ch.f.'s with the finite first moments due to Korolev and Shevtsova [8]

$$|\mathbb{E}e^{itX} - \mathbb{E}e^{itY}| \leq 2 \sin\left(\frac{|t|}{2} L_1(X, Y) \wedge \frac{\pi}{2}\right), \quad \mathbb{E}|X|, \mathbb{E}|Y| < \infty, \quad t \in \mathbb{R}, \quad (8)$$

$a \wedge b \equiv \min\{a, b\}$, $a, b \in \mathbb{R}$, it is not difficult to conclude that

$$\begin{aligned} r(t) &\equiv |f(t) - e^{-t^2/2}| \leq e^{-t^2/2} \int_0^{|t|} |\mathbb{E}e^{iuX} - \mathbb{E}e^{iuX^{(z)}}| u e^{u^2/2} du \leq \\ &\leq 2e^{-t^2/2} \int_0^{|t|} \sin\left(\frac{u}{2} L_1(X, X^{(z)}) \wedge \frac{\pi}{2}\right) u e^{u^2/2} du, \quad t \in \mathbb{R}. \end{aligned}$$

Finally, applying inequality (3) to estimate $L_1(X, X^{(z)})$ one obtain

$$r(t) \leq 2e^{-t^2/2} \int_0^{|t|} \sin\left(\frac{\beta_3 u}{4} \wedge \frac{\pi}{2}\right) u e^{u^2/2} du, \quad t \in \mathbb{R}, \quad (9)$$

for any r.v. X with $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 = \beta_3 < \infty$. Estimate (9) is exact as $t \rightarrow 0$, since it implies that for all $t \geq 0$ and $\beta_3 \geq 1$ such that $\beta_3 t \leq \pi/2$ we have

$$r(t) \leq 2 \int_0^t u \sin\left(\frac{\beta_3 u}{4}\right) du = \frac{32}{\beta_3^2} \left(\sin \frac{\beta_3 t}{4} - \frac{\beta_3 t}{4} \cos \frac{\beta_3 t}{4} \right) \sim \frac{\beta_3 t^3}{6}, \quad t \rightarrow 0,$$

with the least possible factor $1/6$, and is always sharper than the power-type estimate $r(t) \leq \beta_3 |t|^3/6$ especially for moderate (separated from zero) values of $\beta_3 |t|$. Note that $\beta_3 |t|$ can be separated from zero for large enough values of $\beta_3 \geq 1$ even if $|t|$ is small. Thus, the estimates for $r(t)$ of an integral (9)-type form play an important role in the construction of the least possible upper moment-type bounds of the accuracy of the normal approximation which should be uniform in some classes of distributions, especially if in these classes extremal distributions have large third absolute moments. This situation is typical, for example, for the problem of optimization of the absolute constants in the Berry–Esseen-type inequalities with an improved structure (see [9, 10, 8, 12, 16] where a smoothing inequality is applied with the subsequent estimation of the difference $|f_n(t) - e^{-t^2/2}|$, $f_n(t)$ being the ch.f. of the normalized sum of independent random variables, in terms of the difference $|f(t) - e^{-t^2/2}|$, $f(t)$ being the ch.f. of a single r.v.) and in its non-uniform analogues for sums of independent r.v.'s that use the Berry–Esseen inequality with an improved structure (see [1, 15, 6]), as well as in the moment-type estimates of the rate of convergence in limit theorems for compound and mixed compound Poisson distributions (where $\beta_3 \rightarrow \infty$, see [11, 8, 14]) which use the Berry–Esseen inequality with an improved structure as well.

The above reasoning suggests that for the moderate values of $\beta_3 |t|$, estimates for $r(t)$ in the twice-integrated form might be even sharper than estimates in the once-integrated form like (9). Since the ch.f. $f(t)$ is supposed to be differentiable at least twice, it is possible to continue (7) as

$$\begin{aligned} f(t) - e^{-t^2/2} &= e^{-t^2/2} \int_0^t e^{u^2/2} \int_0^u (f'(s) + sf(s))' ds du = \\ &= e^{-t^2/2} \int_0^t e^{u^2/2} \int_0^u (f''(s) + f(s) + sf'(s)) ds du = \\ &= e^{-t^2/2} \int_0^t e^{u^2/2} \int_0^u \left(\mathbb{E}e^{isX} - \mathbb{E}e^{isX^*} + sf'(s) \right) ds du, \end{aligned}$$

or as

$$\begin{aligned} f(t) - e^{-t^2/2} &= e^{-t^2/2} \int_0^t \int_0^u (f(s)e^{s^2/2} - 1)'' ds du = \\ &= e^{-t^2/2} \int_0^t \int_0^u \left(f''(s) + f(s) + sf'(s) + s(f'(s) + sf(s)) \right) e^{s^2/2} ds du = \\ &= e^{-t^2/2} \int_0^t \int_0^u \left(\mathbb{E}e^{isX} - \mathbb{E}e^{isX^*} + sf'(s) + s^2(\mathbb{E}e^{isX} - \mathbb{E}e^{isX^{(z)}}) \right) e^{s^2/2} ds du. \end{aligned}$$

Note that the second estimate contains the additional term $s^2(\mathbb{E}e^{isX} - \mathbb{E}e^{isX^{(z)}})$, but the factor $e^{s^2/2}$ does not exceed the analogous factor $e^{u^2/2}$ in the first one. However, for all $0 \leq s \leq t$ this additional term satisfies

$$g_3(s) \equiv s^2 |\mathbb{E}e^{isX} - \mathbb{E}e^{isX^{(z)}}| \leq 2s^2 \leq 2t^2 = O(t^2), \quad t \rightarrow 0,$$

(actually, an even sharper estimate can be obtained, if inequalities (8) and (3) are used). If $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, $\beta_3 = \mathbf{E}|X|^3 < \infty$, then $|f'(s) + s| \leq \beta_3 s^2/2$, and thus for all $0 \leq s \leq t$ and bounded $\beta_3 t$ we have

$$g_2(s) = s|f'(s)| \leq s^2 + \frac{\beta_3 s^3}{2} \leq t^2 + \frac{\beta_3 t^3}{2} = O(t^2), \quad t \rightarrow 0.$$

So,

$$\sup_{0 \leq s \leq t} (g_2(s) + g_3(s)) = O(t^2), \quad t \rightarrow 0,$$

while the first term

$$g_1(s) = |\mathbf{E}e^{isX} - \mathbf{E}e^{isX^*}| = |f(s) + f''(s)|$$

should be equivalent to $\beta_3 s$ as $s \rightarrow 0+$ in order that the final integrated estimate should have the exact order $\beta_3 |t|^3/6$ as $t \rightarrow 0$. Thus, it is $g_1(s)$ that determines the behavior of the final integral estimate for small values of s , and the problem of construction of the least possible bound for $g_1(s)$ is very important. Theorem 1 gives an opportunity to construct such a bound. Namely, the following corollaries hold.

COROLLARY 1. *Let X be a r.v. with the ch.f. $f(t)$ and $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, $\beta_3 \equiv \mathbf{E}|X|^3 < \infty$. Then for all $t \in \mathbb{R}$*

$$|f(t) + f''(t)| \leq 2 \sin\left(\frac{\beta_3 |t|}{2} \wedge \frac{\pi}{2}\right).$$

COROLLARY 2. *Let X be a r.v. with the ch.f. $f(t)$ and $\mathbf{E}X = 0$, $\mathbf{E}X^2 = 1$, $\beta_3 \equiv \mathbf{E}|X|^3 < \infty$. Then for all $t \in \mathbb{R}$*

$$\begin{aligned} |f(t) - e^{-t^2/2}| &\leq e^{-t^2/2} \int_0^{|t|} \min \left\{ e^{u^2/2} \left(2 \int_0^u \sin\left(\frac{\beta_3 s}{2} \wedge \frac{\pi}{2}\right) ds + \frac{u^3}{3} + \frac{\beta_3 u^4}{8} \right), \right. \\ &\quad \left. \int_0^u \left(2 \sin\left(\frac{\beta_3 s}{2} \wedge \frac{\pi}{2}\right) + 2s^2 \sin\left(\frac{\beta_3 s}{4} \wedge \frac{\pi}{2}\right) + s^2 + \frac{\beta_3 s^3}{2} \right) e^{s^2/2} ds \right\} du. \end{aligned}$$

Note that, as $t \rightarrow 0+$, the r.-h. sides of the inequalities presented in corollary 2 are equivalent to

$$\begin{aligned} 2 \int_0^t \int_0^u \sin\left(\frac{\beta_3 s}{2}\right) ds du &= \frac{8}{\beta_3^2} \left(\frac{\beta_3 t}{2} - \sin \frac{\beta_3 t}{2} \right) < \\ &< \frac{32}{\beta_3^2} \left(\sin \frac{\beta_3 t}{4} - \frac{\beta_3 t}{4} \cos \frac{\beta_3 t}{4} \right) = 2 \int_0^t u \sin\left(\frac{\beta_3 u}{4}\right) du, \end{aligned}$$

provided that $0 \leq \beta_3 t \leq \pi$.

So, corollary 2 plays an important role in estimation of the rate of convergence in limit theorems for sums of independent random variables mentioned above. However, particular application of corollary 2 is the subject of a separate investigation and will be published elsewhere.

3 Proof of theorem 1

The proof consists of three parts: first, we reduce the problem to the class of discrete distributions concentrated in a finite number of points; second, we will demonstrate that the extremal distribution has at most 3 atoms; and finally, we consider two- and tree-point distributions.

1. Reduction to the class of distributions concentrated in a finite number of points. Let X be a r.v. defined on some probability space $(\Omega, \mathcal{A}, \mathbf{P})$. Assume that $\mathbf{E}|X|^3 < \infty$. On the same probability space, construct a sequence of r.v.'s $\{X_n\}_{n \geq 1}$ such that:

- 1) for each n , X_n takes at most a finite number of values,
- 2) $\mathbf{E}X_n = 0$, $\mathbf{E}X_n^2 = 1$,
- 3) $L_1(X_n, (X_n)^*) \rightarrow L_1(X, X^*)$ and $\mathbf{E}|X_n|^3 \rightarrow \mathbf{E}|X|^3$ as $n \rightarrow \infty$.

Denote the indicator function of a set A by $\mathbb{I}(A)$. For $n = 1, 2, \dots$ let

$$Y_n = \sum_{k=1}^{n2^n} \frac{k-1}{2^n} \mathbb{I}\left(\frac{k-1}{2^n} \leq X \leq \frac{k}{2^n}\right) + \sum_{k=-n2^n}^{-1} \frac{k}{2^n} \mathbb{I}\left(\frac{k}{2^n} \leq X \leq \frac{k+1}{2^n}\right).$$

Then, evidently, Y_n takes no more than $n2^{n+1}$ values, $|Y_n| \leq n$. Moreover, $Y_n \rightarrow X$ a.s., $|Y_n| \leq |X|$, $\mathbf{E}|X|^3 < \infty$, hence, by virtue of the Lebesgue dominated convergence theorem, $\mathbf{E}|Y_n|^3 \rightarrow \mathbf{E}|X|^3$, $\mathbf{E}Y_n^2 \rightarrow \mathbf{E}X^2 = 1$, $\mathbf{E}Y_n \rightarrow \mathbf{E}X = 0$, and, consequently, $\mathbf{Var}Y_n \rightarrow 1$ as $n \rightarrow \infty$. So, there exists a number $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$ we have $|\mathbf{E}Y_n| \leq 1$ and $\mathbf{Var}Y_n \geq 1/4$. Now let

$$X_n = \frac{Y_n - \mathbf{E}Y_n}{\sqrt{\mathbf{Var}Y_n}}, \quad n = 1, 2, \dots$$

Then X_n takes at most a finite ($\leq n2^{n+1}$) number of values, $\mathbf{E}X_n = 0$, $\mathbf{E}X_n^2 = 1$, $X_n \rightarrow X$ a.s. and for all $n \geq n_0$

$$|X_n| \leq \frac{|Y_n| + |\mathbf{E}Y_n|}{\sqrt{\mathbf{Var}Y_n}} \leq 2(|X| + 1) \equiv \eta, \quad \mathbf{E}\eta^3 < \infty,$$

hence $\mathbf{E}|X_n - X|^3 \rightarrow 0$ and $\mathbf{E}|X_n|^3 \rightarrow \mathbf{E}|X|^3$ as $n \rightarrow \infty$. So, it remains to show that $L_1(X_n, (X_n)^*) \rightarrow L_1(X, X^*)$.

Using the triangle inequality we have

$$L_1(X, X^*) \leq L_1(X, X_n) + L_1(X_n, (X_n)^*) + L_1((X_n)^*, X^*),$$

$$L_1(X_n, (X_n)^*) \leq L_1(X_n, X) + L_1(X, X^*) + L_1(X^*, (X_n)^*),$$

which is equivalent to

$$|L_1(X_n, (X_n)^*) - L_1(X, X^*)| \leq L_1(X, X_n) + L_1(X^*, (X_n)^*). \quad (10)$$

The first term in (10) tends to zero since

$$L_1(X, X_n) \leq \mathbf{E}|X - X_n| \leq (\mathbf{E}|X - X_n|^3)^{1/3} \rightarrow 0, \quad n \rightarrow \infty.$$

Consider the second term $L_1(X^*, (X_n)^*)$. As is well known, the L_1 -metric can be represented in terms of the ζ_1 -metric as

$$L_1(X, Y) = \zeta_1(X, Y) \equiv \sup\{|\mathbf{E}f(X) - \mathbf{E}f(Y)| : f \in \mathcal{F}_1\}, \quad \mathbf{E}|X| < \infty, \mathbf{E}|Y| < \infty,$$

where \mathcal{F}_1 is the set of all real-valued functions on \mathbb{R} such that $\sup_{x \neq y} |f(x) - f(y)|/|x - y| \leq 1$. Therefore, using the elementary property of the shape-biased d.f. $dF^*(x) = x^2 dF(x)$, $x \in \mathbb{R}$, we conclude that

$$L_1(X^*, (X_n)^*) = \zeta_1(X^*, (X_n)^*) = \sup_{f \in \mathcal{F}_1} |\mathbb{E}X^* - \mathbb{E}(X_n)^*| = \sup_{f \in \mathcal{F}_1} |\mathbb{E}X^2 f(X) - \mathbb{E}X_n^2 f(X_n)|.$$

Fix any function $f \in \mathcal{F}_1$. Without loss of generality it can be assumed that $f(0) = 0$, since $\mathbb{E}X_n^2 = \mathbb{E}X^2 = 1$ and

$$\mathbb{E}X^2 f(X) - \mathbb{E}X_n^2 f(X_n) = \mathbb{E}X^2(f(X) - f(0)) - \mathbb{E}X_n^2(f(X_n) - f(0)).$$

Then for this function f we have $|f(x) - f(y)| \leq |x - y|$, $x, y \in \mathbb{R}$, in particular, if $y = 0$, then $|f(x)| \leq |x|$, and hence

$$\begin{aligned} \frac{|x^2 f(x) - y^2 f(y)|}{|x - y|} &\leq \frac{x^2 |f(x) - f(y)|}{|x - y|} + |f(y)| \cdot \frac{|x^2 - y^2|}{|x - y|} \leq \\ &\leq x^2 + |y|(|x| + |y|) = x^2 + y^2 + |xy| \leq (|x| + |y|)^2, \quad x \neq y. \end{aligned}$$

So, by the Hölder inequality we obtain

$$\begin{aligned} \zeta_1(X^*, (X_n)^*) &\leq \sup \{ \mathbb{E}|X^2 f(X) - X_n^2 f(X_n)| \mathbb{I}(X_n \neq X) : f \in \mathcal{F}_1, f(0) = 0 \} \leq \\ &\leq \mathbb{E}|X - X_n| \mathbb{I}(X_n \neq X) (|X_n| + |X|)^2 \leq \mathbb{E}|X - X_n| (|X_n| + |X|)^2 \leq \\ &\leq (\mathbb{E}|X - X_n|^3)^{1/3} (\mathbb{E}(|X_n| + |X|)^3)^{2/3} \rightarrow 0, \quad n \rightarrow \infty, \end{aligned}$$

since $\lim_{n \rightarrow \infty} \mathbb{E}|X - X_n|^3 \rightarrow 0$ and $\mathbb{E}(|X_n| + |X|)^3 < \infty$ (the expectations $\mathbb{E}|X_n|^3$, $\mathbb{E}|X|^3$ being finite).

Thus, the second term $L_1(X^*, (X_n)^*)$ in (10) also tends to zero as $n \rightarrow \infty$. The sequence X_n satisfies all the conditions 1)–3) and in further reasoning, to prove the theorem it suffices to consider only discrete distributions concentrated in a finite number of points.

2. Reduction to the class of distributions concentrated in at most three points.

Since for any function $f \in \mathcal{F}_1$ we also have $(-f) \in \mathcal{F}_1$ and $\mathbb{E}f(X^*) = \mathbb{E}X^2 f(X)$,

$$L_1(X, X^*) = \zeta_1(X, X^*) = \sup_{f \in \mathcal{F}_1} (\mathbb{E}f(X) - \mathbb{E}f(X^*)) = \sup_{f \in \mathcal{F}_1} \mathbb{E}(1 - X^2)f(X).$$

For $f \in \mathcal{F}_1$ denote

$$J(X, f) = \mathbb{E}((1 - X^2)f(X) - |X|^3).$$

Then

$$L_1(X, X^*) - \mathbb{E}|X|^3 = \sup_{f \in \mathcal{F}_1} J(X, f),$$

and the statement of the theorem is equivalent to $\sup J(X, f) = 0$, where the supremum is taken over all $f \in \mathcal{F}_1$ and all r.v.'s X taking a finite number of values and satisfying the conditions $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$, $\mathbb{E}|X|^3 < \infty$. As it follows from the results of W. Hoeffding [7], the supremum of a linear (with respect to the d.f. $F(x)$) functional $J(X, f)$ over all distribution functions $F(x)$ of a r.v. X concentrated in a finite number of points and satisfying two linear (with respect to the d.f. $F(x)$) equality-type conditions $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$ is attained at the distributions concentrated in at most three points.

3. Checking two- and three-point distributions. Here we will use one more representation of the L_1 -metric in terms of the mean-metric:

$$L_1(X, Y) = \varkappa(X, Y) \equiv \int_{-\infty}^{\infty} |\mathbb{P}(X < u) - \mathbb{P}(Y < u)| du, \quad \mathbb{E}|X| < \infty, \quad \mathbb{E}|Y| < \infty.$$

In particular, with $Y = X^*$ and $\mathbb{P}(X^* < u) = \mathbb{E}X^2\mathbb{I}(X < u)$, we have

$$L_1(X, X^*) = \int_{-\infty}^{\infty} |F(u) - \mathbb{E}X^2\mathbb{I}(X < u)| du.$$

Let a r.v. X take two values and satisfy the conditions $\mathbb{E}X = 0$, $\mathbb{E}X^2 = 1$. Then its distribution should necessarily have the form

$$\mathbb{P}\left(X = \sqrt{q/p}\right) = p = 1 - \mathbb{P}\left(X = -\sqrt{p/q}\right), \quad q = 1 - p \in (0, 1).$$

It is easy to see that $\mathbb{E}X^3 = (q - p)/\sqrt{pq}$, $\mathbb{E}|X|^3 = (p^2 + q^2)/\sqrt{pq}$. Then

$$\mathbb{E}X^2\mathbb{I}(X < u) = \begin{cases} 0, & u \leq -\sqrt{p/q}, \\ p, & -\sqrt{p/q} < u \leq \sqrt{q/p}, \\ 1, & \sqrt{q/p} < u, \end{cases}$$

$$\mathbb{E}X^2\mathbb{I}(X < u) - F(u) = (p - q)\mathbb{I}(-\sqrt{p/q} < u \leq \sqrt{q/p}),$$

and hence

$$L_1(X, X^*) = \int_{-\infty}^{\infty} |p - q|\mathbb{I}(-\sqrt{p/q} < u \leq \sqrt{q/p}) du = \frac{|p - q|}{\sqrt{pq}} = |\mathbb{E}X^3| \leq \mathbb{E}|X|^3$$

by virtue of the Jensen inequality, thus, the statement of the theorem holds. Moreover, for any $\varepsilon > 0$

$$\frac{L_1(X, X^*)}{\mathbb{E}|X|^3} = \frac{|1 - 2p|}{1 - 2p + 2p^2} \geq 1 - 2p > 1 - \varepsilon$$

for all $0 < p < \varepsilon/2$.

Now consider a r.v. X taking exactly three values. Note that

$$\sup \left\{ \frac{L_1(X, X^*)}{\mathbb{E}|X|^3} : \mathbb{E}X = 0, \mathbb{E}X^2 = 1 \right\} = \sup \left\{ \frac{L_1(X, X^*)\sigma^2}{\mathbb{E}|X|^3} : \sigma > 0, \mathbb{E}X = 0, \mathbb{E}X^2 = \sigma^2 \right\},$$

where the supremums are taken over three-point distributions of a r.v. X . Let X take the three values, x, y, z with the probabilities $p, q, r > 0$ respectively, $p + q + r = 1$. Without loss of generality one can assume that $x < y \leq 0 < z$. From the conditions $\mathbb{E}X = 0$, $\mathbb{E}X^2 = \sigma^2$ we find that

$$p = \frac{\sigma^2 + yz}{(z - x)(y - x)}, \quad q = -\frac{\sigma^2 + xz}{(z - y)(y - x)}, \quad r = \frac{\sigma^2 + xy}{(z - x)(z - y)}, \quad -yz < \sigma^2 < -xz.$$

For all $u \in \mathbb{R}$ we have

$$\mathbb{E}X^2\mathbb{I}(X < u) = \begin{cases} 0, & u \leq x, \\ px^2, & x < u \leq y, \\ px^2 + qy^2, & y < u \leq z, \\ \sigma^2, & z < u, \end{cases}$$

$$\sigma^{-2}\mathbb{E}X^2\mathbb{I}(X < u) - F(u) = \begin{cases} 0, & u \leq x, \\ p(x^2/\sigma^2 - 1), & x < u \leq y, \\ (px^2 + qy^2)/\sigma^2 - p - q, & y < u \leq z, \\ 0, & z < u. \end{cases}$$

Noticing that $(px^2 + qy^2)/\sigma^2 - p - q = (\sigma^2 - rz^2)/\sigma^2 - 1 + r = r(1 - z^2/\sigma^2)$, we obtain

$$L_1(X, X^*) = \int_{-\infty}^{\infty} |F(u) - \sigma^{-2}\mathbb{E}X^2\mathbb{I}(X < u)| du = p \left| \frac{x^2}{\sigma^2} - 1 \right| (y - x) + r \left| 1 - \frac{z^2}{\sigma^2} \right| (z - y).$$

Consider the function

$$\begin{aligned} L_1(X, X^*)\sigma^2 - \mathbb{E}|X|^3 &= p(y - x)|x^2 - \sigma^2| + r(z - y)|z^2 - \sigma^2| + px^3 + qy^3 - rz^3 = \\ &= \frac{1}{z - x} \left(|x^2 - \sigma^2|(\sigma^2 + yz) + |z^2 - \sigma^2|(\sigma^2 + xy) - \frac{2z^3(\sigma^2 + xz)}{z - y} + \right. \\ &\quad \left. + \sigma^2(z^2 - x^2 - xy + yz) + xyz(z - x) + 2xz^3 \right) \equiv g(x, y, z, \sigma^2). \end{aligned}$$

The statement of the theorem is equivalent to $\sup g(x, y, z, \sigma^2) = 0$, where the supremum is taken over all $\sigma^2 > 0$, $x < y \leq 0 < z$ such that $-yz < \sigma^2 < -xz$. Note that it suffices to consider only $\sigma^2 < \max\{x^2, z^2\}$, since the opposite inequality (with $q > 0$) implies that $\mathbb{E}X^2 < \sigma^2$. So, there are only three possibilities: 1) $0 < \sigma^2 < \min\{x^2, z^2\}$, 2) $x^2 \leq \sigma^2 < z^2$, 3) $z^2 \leq \sigma^2 < x^2$. Opening the modules, we notice that $g(x, y, z, \sigma^2)$ is a parabola with respect to σ^2 on each of the intervals specified above. Consider the behavior of $g(x, y, z, \sigma^2)$ on each of these intervals.

1. $0 < \sigma^2 < \min\{x^2, z^2\}$, then necessarily $\sigma^2 < -xz$ and

$$g(x, y, z, \sigma^2) = -\frac{2(\sigma^2 + xy)(yz^2 + \sigma^2(z - y))}{(z - y)(z - x)}.$$

The coefficient $-2/(z - x)$ at σ^4 is negative, thus the branches of this parabola with respect to σ^2 look down and the maximum value of the function $g(z, y, z, \sigma^2)$ within the interval $0 < \sigma^2 < \min\{x^2, z^2\}$ is attained either at the vertex

$$\sigma_*^2 = -\frac{y(z^2 + xz - xy)}{2(z - y)},$$

if $\sigma_*^2 > -yz$, or at the point $\sigma^2 \rightarrow -yz + 0$, if $\sigma_*^2 \leq -yz$. We have

$$\sigma_*^2 + yz = \frac{y(z(z - x) - y(2z - x))}{2(z - y)} \leq 0,$$

since $x < y \leq 0 < z$, with the equality attained if and only if $y = 0$. Thus, the supremum is attained as $\sigma^2 \rightarrow -yz + 0$, which implies that $p \rightarrow 0$ and reduces the problem to checking two-point distributions considered above.

2. $z^2 \leq \sigma^2 < x^2$, then

$$g(x, y, z, \sigma^2) = -\frac{2z^3(\sigma^2 + xy)}{(z - x)(z - y)} = -2rz^3 < 0$$

by virtue of the conditions $r > 0$, $z > 0$.

3. $x^2 \leq \sigma^2 < z^2$, then the function

$$g(x, y, z, \sigma^2) = 2 \frac{-\sigma^2(x^2(z-y) + y^2(z-x) + xyz) + xyz(xy - xz - yz)}{(z-x)(z-y)}$$

is linear and decreases monotonically in σ^2 , since $x^2(z-y) + y^2(z-x) + xyz > 0$. Thus, if $x^2 \leq -yz$, then the supremum of $g(x, y, z, \sigma^2)$ is supplied by $\sigma^2 \rightarrow -yz + 0$, which reduces the problem to checking two-point distributions considered above. If $x^2 > -yz$, then the supremum of $g(x, y, z, \sigma^2)$ is attained at $\sigma^2 = x^2$. With this value of σ^2 we have

$$g(x, y, z, x^2) = -\frac{2x(x+y)(x^2z - yx^2 + yz^2)}{(z-x)(z-y)}.$$

$$\frac{\partial}{\partial y} g(x, y, z, x^2) = -\frac{2x(x+z)(yz(2z-y) + x(z-y)^2)}{(z-x)(z-y)^2} < 0,$$

since $x < -z$ and $x < y \leq 0 < z$. Thus, the supremum of $g(x, y, z, x^2)$ over all y such that $x < y \leq 0$ and $-yz < x^2 = \sigma^2$ is supplied by $y \rightarrow \max\{x, -x^2/z\} + 0 = -x^2/z + 0$, i. e., $p \rightarrow 0$, which reduces the problem to checking two-point distributions considered above. Thus, the theorem is completely proved.

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